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Exact Spin and Pseudo-Spin Symmetric Solutions of the Dirac-Kratzer Problem with a tensor potential via Laplace Transform Approach

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Exact bound state solutions of the Dirac equation for the Kratzer potential in the presence of a tensor potential are studied by using the Laplace transform approach for the cases of spin- and pseudo-spin symmetry. The energy spectra is obtained in the closed form for the relativistic as well as non-relativistic cases including the Coulomb potential. It is seen that our analytical results are in agrement with the ones given in literature. The numerical results are also given in a table for different parameter values.

1 Introduction

The analytical solvable problems within the framework of quantum mechanics are very restricted. In this manner, it could be interesting to find the exact analytical solutions of the Kratzer potential in the relativistic domain. We intend to study the radial Dirac equation for the potential having the form

$$V(r) = -2Da\left(\frac{1}{r} - \frac{a/2}{r^2}\right), \tag{1}$$

where $D > 0$ and $a > 0$. The detailed analysis of the bound states of this potential including a third parameter q in the non-relativistic domain are placed in literature [1–3] where used the WKB approximation and the semiclassical quantization procedure with the help of the Langer transformation. The potential form including the parameter q provides the extension of the spectra of the usual Kratzer potential [4, 5].

We find the exact bound state solutions and the corresponding upper- and lower-spinor of the radial Dirac equation in the presence of a tensor interaction within the framework of the Laplace transformation (LT) scheme [6–9] for the cases of spin symmetry, $S(r) = +V(r)$, and pseudo-spin symmetry, $S(r) = -V(r)$. The studying the solutions of the Dirac equation for the spin and pseudo-spin symmetric cases has been received a great attention in literature [10–12].

The organization of this work is as follows. In Section 2, we find the energy spectrum and corresponding wave functions of the Kratzer potential [13–16]

for the spin and pseudo-spin symmetric cases which have been studied in details for the above potential without the tensor interaction in Ref. [17]. We also study the energy spectra of the Coulomb potential and see that the result obtained for the Coulomb problem is the same with the one obtained in Ref. [18]. At the end, we give the bound state spectrum of the above potentials for the non-relativistic limit where the results are consistent with the ones obtained in the literature.

2 Bound States

Dirac equation for the scalar $S(r)$ and vector $V(r)$ potentials with a tensor interaction $U(r)$ is ($\hbar = c = 1$) [19]

$$\left\{\vec{\alpha}.\vec{p} + \beta\left[m_0 + S(r)\right] + V(r) - i\beta.\hat{r}U(r) - E_{n\kappa}\right\}\Psi(\vec{r}) = 0\,, \tag{2}$$

where $\vec{p}$ is the momentum operator, $E_{n\kappa}$ is the relativistic energy of the particle with the rest mass m_0 and $\vec{\alpha}$ written in terms of Pauli matrices and β are 4×4 Dirac matrices, respectively. The energy of the particle $E_{n\kappa}$ depends on the radial quantum number n and spin-orbit quantum number κ. Inserting the eigenfunctions in terms of the spherical harmonic functions $Y_{\kappa}(\theta,\phi)$ and $Y_{-\kappa}(\theta,\phi)$

$$\Psi_{n\kappa}(r) = \frac{1}{r}\left(\begin{array}{c} F(r)Y_{\kappa}(\theta,\phi) \\ iG(r)Y_{-\kappa}(\theta,\phi)\end{array}\right), \tag{3}$$

gives the following coupled differential equations for the upper-spinor $F(r)$ and lower-spinor $G(r)$

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F(r) = \left[m_0 + E_{n\kappa} + S(r) - V(r)\right]G(r)\,, \tag{4a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G(r) = \left[m_0 - E_{n\kappa} + S(r) + V(r)\right]F(r)\,. \tag{4b}$$

Substituting Eq. (4a) into Eq. (4b), we obtain a second-order differential equation

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - \left[\frac{d}{dr} - \frac{2\kappa}{r} + U(r)\right]U(r) - \frac{\frac{d}{dr}[S(r)-V(r)][\frac{d}{dr} + \frac{\kappa}{r} - U(r)]}{m_0 + E_{n\kappa} + S(r) - V(r)}\right]F(r)$$
$$- \left[m_0^2 - E_{n\kappa}^2 + 2\left(m_0S(r) + E_{n\kappa}V(r)\right) + S^2(r) - V^2(r)\right]F(r) = 0\,. \tag{5}$$

We consider the case where the scalar and vector potentials are equal, $S(r) = V(r)$, which corresponds to the spin symmetry of the Dirac equation [10–12]. Inserting Eq. (1) into Eq. (5), taking the tensor interaction as $-C/r$ with a constant C and using a new wave function as $F(r) = \sqrt{r}\,\phi(r)$ gives

$$\frac{d^2\phi(r)}{dr^2} + \frac{1}{r}\frac{d\phi(r)}{r} - \frac{1}{r^2}\left(a_1^2 + a_2^2 r + \epsilon_{n\kappa}^2 r^2\right)\phi(r) = 0\,, \tag{6}$$

where

$$a_1^2 = \kappa(\kappa+1) + C(1+2\kappa+C) + 2Da^2(m_0+E_{n\kappa}) + \frac{1}{4}\,, \tag{7a}$$

$$a_2^2 = -4Da(m_0+E_{n\kappa})\,, \tag{7b}$$

$$\epsilon_{n\kappa}^2 = -(E_{n\kappa}^2 - m_0^2)\,. \tag{7c}$$

Defining a wave function $\phi(r) = r^A f(r)$ with A as a constant and inserting into Eq. (6) leads

$$r\frac{d^2 f(r)}{dr^2} - (2a_1-1)\frac{df(r)}{dr} - \left(a_2^2 + \epsilon_{n\kappa}^2 r\right) f(r) = 0\,, \tag{8}$$

where we set $A = -a_1$ to obtain a finite solution when $r \to \infty$. By using the LT defined as [20]

$$\mathscr{L}\left\{f(r)\right\} = f(t) = \int_0^\infty dr e^{-tr} f(r)\,, \tag{9}$$

we obtain a first-order differential equation from Eq. (8) as

$$\left(t^2 - \epsilon_{n\kappa}^2\right)\frac{df(t)}{dt} + \left[\left(2a_1+1\right)t + a_2^2\right] f(t) = 0\,. \tag{10}$$

It's solution is

$$f(t) \sim \left(t+\epsilon_{n\kappa}\right)^{-(2a_1+1)}\left(\frac{t-\epsilon_{n\kappa}}{t+\epsilon_{n\kappa}}\right)^{-\frac{a_2^2}{2\epsilon_{n\kappa}}-\frac{2a_1+1}{2}}\,. \tag{11}$$

We impose the following condition to obtain a single-valued wave function

$$-\frac{a_2^2}{2\epsilon_{n\kappa}} - \frac{1}{2}(2a_1+1) = n\,.\ (n=0,1,2,3,\ldots) \tag{12}$$

Taking into account this condition and applying a simple series expansion to Eq. (11) gives

$$f(t) \sim \sum_{m=0}^{n}\frac{(-1)^m n!}{(n-m)!m!}(2\epsilon_{n\kappa})^m\left(t+\epsilon_{n\kappa}\right)^{-(2a_1+1)-m}\,, \tag{13}$$

Using the inverse Laplace transformation [20] in Eq. (13) we obtain

$$f(r) \sim r^{2a_1}e^{-\epsilon_{n\kappa}r}\sum_{m=0}^{n}\frac{(-1)^m n!}{(n-m)!m!}\frac{\Gamma(2a_1+1)}{\Gamma(2a_1+1+m)}(2\epsilon_{n\kappa}r)^m\,, \tag{14}$$

and

$$F(r) = Nr^{a_1+\frac{1}{2}}e^{-\epsilon_{n\kappa}r}\,{}_1F_1(-n, 2a_1+1, 2\epsilon_{n\kappa}r)\,, \tag{15}$$

where N is a normalization constant and used the following identity for the hypergeometric functions [21]

$$ {}_1F_1(-n,\sigma,x)=\sum_{p=0}^{n}\frac{(-1)^p n!}{(n-p)!p!}\frac{\Gamma(\sigma)}{\Gamma(\sigma+p)}x^p\,, \tag{16} $$

The relation between the Laguerre polynomials and confluent hypergeometric functions as $L_n^{\eta}(x)=\frac{\Gamma(n+\eta+1)}{n!\Gamma(\eta+1)}\,{}_1F_1(-n,\eta+1,x)$ [21] gives the wave functions as

$$ F(r)=\frac{n!\Gamma(2a_1+1)}{n+2a_1+1}\,r^{a_1+\frac{1}{2}}e^{-\epsilon_{n\kappa}r}L^{(2a_1)}(2\epsilon_{n\kappa}r)\,. \tag{17} $$

Inserting the parameters in Eq. (7) into Eq. (12), we obtain the energy spectra of the Kratzer potential in the presence of a Coulomb-type tensor interaction

$$ E_{n\kappa}=m_0\frac{\left[1+2n+\sqrt{(1+2\kappa)^2+4C(1+2\kappa+C)+8Da^2(E_{n\kappa}+m_0)}\right]^2-16D^2a^2}{\left[1+2n+\sqrt{(1+2\kappa)^2+4C(1+2\kappa+C)+8Da^2(E_{n\kappa}+m_0)}\right]^2+16D^2a^2}\,. \tag{18} $$

which is the same with the ones obtained in Ref. [17] for $C=0$. In Table 1, we present the numerical results for the case where $C\neq 0$ to see the effect of the tensor interaction on the energy spectrum. It is seen that the energy eigenvalues decrease while the strength of the tensor interaction increases. It is interesting to obtain the solution for the Coulomb case, setting the second term in Eq. (1) to zero, where we set the parameters as $Da=Ze^2$ (Ze^2 is the charge of the nucleus) and $\kappa=\ell$ in Eq. (18). This gives in the absence of tensor potential ($\hbar=c=1$)

$$ E_{n\ell}=m_0\left(1-\frac{8Z^2e^4}{(n+\ell+1)^2+4Z^2e^4}\right)\,. \tag{19} $$

It is worth to say that this result is the same with the one obtained for the well-known Dirac-Coulomb problem in Ref. [18].

Now we intend to give the results for the Kratzer potential and Coulomb potential in the non-relativistic limit.

2.1 Non-Relativistic Kratzer Limit

In order to obtain the energy eigenvalues of the Kratzer potential in the non-relativistic limit for $C=0$, we set $E_{n\kappa}-m_0c^2\simeq E_{n\ell}$ and $E_{n\kappa}+m_0c^2\simeq 2m_0c^2$. Using this assumptions and with the help of Eq. (7), Eq. (12) gives ($\hbar=c=1$)

$$ E_{n\ell}=-\frac{8D^2a^2m_0}{\left[1+2n+\frac{1}{2}\sqrt{1+4\ell(\ell+1)+16Dm_0a^2}\right]^2}\,. \tag{20} $$

which is exactly same with the result obtained in Ref. [22].

2.2 Non-Relativistic Coulomb Limit

We expand Eq. (19) into a series in terms of the parameter Ze^2 in order to obtain the energy spectrum of the Coulomb potential in the non-relativistic limit in the absence of tensor interaction. So, we find the energy equation for $Ze^2 \ll 1$

$$E_{n\ell} \sim m_0 \left[1 - \frac{8Z^2e^4}{(n+\ell+1)^2} + \frac{32Z^4e^8}{(n+\ell+1)^4} + \ldots \right] . \tag{21}$$

which is in agreement with the results obtained in literature [18].

Finally, we briefly study the case where the scalar and vector potentials are equal magnitude but different sign, *i.e.*, $S(r) = -V(r)$ in which the Dirac equation has pseudo-spin symmetry [10–12]. For convenience for the rest of computation, we give the differential equation satisfying by the lower spinor-component. Following the same steps, we obtain the following

$$\frac{d^2\phi(r)}{dr^2} + \frac{1}{r}\frac{d\phi(r)}{r} - \frac{1}{r^2}\left(A_1^2 + A_2^2 r + \varepsilon_{n\kappa}^2 r^2\right)\phi(r) = 0\,, \tag{22}$$

where

$$A_1^2 = \kappa(\kappa-1) + C(2\kappa + C - 1) - 2Da^2(m_0 - E_{n\kappa}) + \frac{1}{4}\,, \tag{23a}$$

$$A_2^2 = 4Da(m_0 - E_{n\kappa})\,, \tag{23b}$$

$$\varepsilon_{n\kappa}^2 = m_0^2 - E_{n\kappa}^2 = \epsilon_{n\kappa}^2\,. \tag{23c}$$

The wave functions can be written as

$$G(r) = \frac{n!\Gamma(2A_1+1)}{n+2A_1+1}\, r^{A_1+\frac{1}{2}} e^{-\varepsilon_{n\kappa} r} L^{(2A_1)}(2\varepsilon_{n\kappa} r)\,, \tag{24}$$

and the energy eigenvalue equation of the Kratzer potential with a tensor interaction is written as

$$E_{n\kappa} = m_0 \left[1 - \frac{2\left[1 + 2n + \sqrt{(1-2\kappa)^2 + 4C(2\kappa+C-1) + 8Da^2(E_{n\kappa} - m_0c^2)}\right]^2}{\left[1 + 2n + \sqrt{(1-2\kappa)^2 + 4C(2\kappa+C-1) + 8Da^2(E_{n\kappa} - m_0c^2)}\right]^2 + 16D^2a^2} \right] . \tag{25}$$

which is the same with the ones obtained in Ref. [17] $C = 0$. Our numerical results are given for the pseudo-spin symmetric case in Table 1 for $C \neq 0$ and see that the energy eigenvalues increase while the strength of the tensor interaction increases.

3 Conclusion

We have exactly solved the radial Dirac equation for the Kratzer potential in the presence of a tensor interaction term by using the Laplace transform approach. The energy eigenvalues and the corresponding upper- and lower-spinor components of the potential are computed for the cases of the spin and pseudo-spin symmetry. We have also obtained the energy eigenvalues of the Coulomb potential by choosing suitable parameter values. We have also studied the non-relativistic limit for the above potentials.

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Table 1: Energy eigenvalues of the Kratzer potential in the presence of the tensor interaction for the parameter values as $m_0 = 5$ fm^{-1}, $D = 1.25$ fm^{-1}, $a = 0.35$ fm^{-1} [17].

		spin symmetry	
n	κ	$C = 0.25$	$C = 0.50$
0	-2	4.63346	4.57251
	-3	4.79992	4.76888
	-4	4.88031	4.86515
	-5	4.92171	4.91357
1	-2	4.82214	4.80240
	-3	4.88546	4.87236
	-4	4.92329	4.91564
	-5	4.94578	4.94114
		pseudo-spin symmetry	
n	κ	$C = 0.25$	$C = 0.50$
1	-1	-4.46315	
	-2	-4.83578	-4.80233
	-3	-4.90668	-4.89445
	-4	-4.93862	-4.93245
2	-1	-4.75526	
	-2	-4.90136	-4.88615
	-3	-4.93728	-4.93064
	-4	-4.95581	-4.95208